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ESTIMATES OF THE MAXIMUM DISPLACEMENT OF A SOLID IN A HYBRID SYSTEM WITH DRY FRICTION†

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A hybrid system consisting of a supporting solid and an elastic element attached to it is considered. The supporting body is in frictional contact with another body (the base) which executes a specified motion. The problem of estimating the maximum displacement of the supporting body relative to the base is studied for the class of perturbations which are generated by the base. An analytic expression is obtained for this estimate. The various limiting cases in which the value of the estimate is identical to the maximum possible displacement of the supporting body are analysed. © 1997 Elsevier Science Ltd. All rights reserved.

1. We consider a hybrid (discrete–continual) mechanical system of the following type. An elastic element B_e with distributed parameters of length l is attached to a supporting solid B_0 of mass m. The solid is in frictional contact with another solid (the base) which executes a specified motion with respect to a certain inertial reference system. Motions of the hybrid system along the ξ axis (Fig. 1) are considered.

Assuming that the deformation of the elastic body is sufficiently small and that the linear density ρ and the compression stiffness σ are constant, we write the equation of state of the elastic body

$$\rho\ddot{u} = \sigma u'' - \rho(\ddot{\xi} + \ddot{y}), \quad 0 < x < l \tag{1.1}$$

Here, u is the relative elastic displacement of a section x, $\ddot{\xi}(t)$ is the acceleration of the base and $\ddot{y}$ is the acceleration of the load-bearing body. We shall assume that the left end of the elastic body is clamped and that the right end is free

$$u(t,0) = u'(t,l) = 0, \quad t \geqslant 0$$

We also assume that, at the initial instant $t = 0$ the elastic body is at rest relative to the supporting body

$$u(0,x) = \dot{u}(0,x) = 0, \quad 0 \leqslant x \leqslant l$$

We write the equation of motion for the supporting body in the form of an integro-differential equation which describes the motion of the centre of mass of the hybrid system (the elastic body plus the supporting solid) relative to the base

$$m(\ddot{y} + \ddot{\xi}) + \int_0^l \rho[\ddot{u}(t,x) + \ddot{y} + \ddot{\xi}]dx = -F(\dot{y}) \tag{1.2}$$

$$F = \begin{cases} F_0 \operatorname{sign} \dot{y}, & \dot{y} \neq 0 \\ q, & \dot{y} = 0,\ |q| \leqslant F_0,\ q = -m\ddot{\xi} - \int_0^l \rho[\ddot{u}(t,x) + \ddot{y} + \ddot{\xi}]dx \\ F_0 \operatorname{sign} q, & \dot{x} = 0,\ |q| > F_0 \end{cases}$$

where the function $F(\dot{y})$ determines the force due to dry friction which acts between the base and the supporting body, and the constant F_0 determines the maximum static friction force.

We shall assume that the supporting body is at rest relative to the base at the initial instant: $y(0) = \dot{y}(0) = 0$ and that the function $(\ddot{\xi}(t))$, which defines the acceleration of the base in the inertial frame of reference is piecewise continuous and satisfies the condition

$$\int_0^\infty |\ddot{\xi}(t)| dt \leqslant J_0 \tag{1.3}$$

We shall assign any form of a motion of the base which satisfies the conditions which have been given to the class D and use the notation $v(t) = -\ddot{\xi}(t)$. We shall henceforth call the functions $v(t)$, which also belong to class D,

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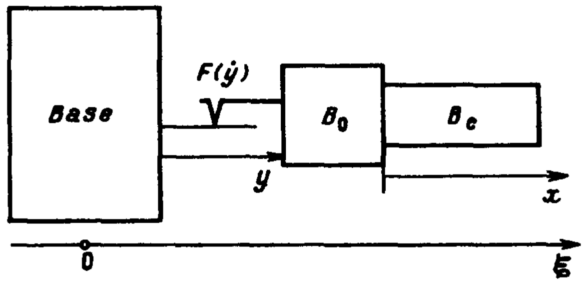

Fig. 1.

perturbations. In the class of functions D, we introduce a functional that characterizes the maximum displacement of the supporting body

$$S[\upsilon(\cdot)] = \sup_{t\in[0,\infty)} |y(t;\upsilon)|$$

where $y(t, \upsilon)$, $u(t; x; \upsilon)$ is the solution of problem (1.1), (1.2) with the above-mentioned initial and boundary conditions which correspond to a specified perturbation $\upsilon(t)$.

From the theoretical and applied points of view (in particular, in problems of vibration insulation) it is important to know the maximum possible displacement of the supporting body under the action of perturbations from class D

$$S_0 = \sup_{\upsilon(\cdot)\in D} S[\upsilon(\cdot)]$$

Problems of this type, but for the simpler case when there is no elastic body, were considered previously in [2–4]. The presence of an elastic body creates serious difficulties in finding the quantity S_0 and makes it impossible to use the methods previously described [3, 4].

We shall confine ourselves to finding "reasonable" upper estimates of the magnitude of S_0. Here, by the term "reasonable", we mean those estimates which give the exact value of S_0 in the limiting cases which have been thoroughly studied: the mass of the elastic body, which is equal to ρl, is infinitesimal compared with the mass of the supporting body m and the stiffness of the elastic body is so large that it can be treated as an absolutely rigid body.

2. Before constructing estimates of the magnitude of the maximum possible deviation S_0, we shall carry out a number of preliminary estimates. We introduce the function

$$E(t;\upsilon) = \frac{m\dot{y}^2(t;\upsilon)}{2} + \frac{1}{2}\int_0^l \rho\{[\dot{u}(t,x;\upsilon) + \dot{y}(t,\upsilon)]^2 + \sigma[u'(t,x;\upsilon)]^2\}dx$$

This function defines the total mechanical energy of the hybrid system in its motion relative to the base. We then find the derivative of $E(t; \upsilon)$ with respect to t from (1.1) and (1.2), omitting, for brevity, the arguments of the functions $u(t, x; \upsilon)$, $y(t; \upsilon)$ where this does not give rise to misunderstanding. We obtain

$$\frac{dE}{dt} = m\dot{y}\ddot{y} + \int_0^l [\rho(\dot{u}+\dot{y})(\ddot{u}+\ddot{y}) + \sigma u'\dot{u}']dx = -F(\dot{y})\dot{y} + \rho l\dot{y}\upsilon(t) + m\upsilon(t)\dot{y} + \int_0^l [\rho\dot{u}(\ddot{u}+\ddot{y}) + \sigma u'\dot{u}']dx$$

On integrating $\sigma u'\dot{u}$ by parts, taking account of the boundary conditions and Eq. (1.1), we have

$$dE/dt = -F(\dot{y})\dot{y} + M\upsilon(t)\dot{y}_c$$
$$\dot{y}_c = \frac{1}{M}[m\dot{y} + \rho\int_0^l(\dot{u}+\dot{y})dx], \quad M = m + \rho l \tag{2.1}$$

($\dot{y}_c$ is the velocity of the centre of mass of the hybrid system).

Using the definition of the dry friction force, we have $F(\dot{y})\dot{y} \geqslant 0$. Consequently, the estimate

$$dE/dt \leqslant M|\upsilon(t)||\dot{y}_c| \tag{2.2}$$

holds.

The inequality

$$K = \frac{m\dot{y}^2}{2} + \frac{\rho}{2}\int_0^l (\dot{u}+\dot{y})^2 dx \geq \frac{M\dot{y}_c^2}{2} \tag{2.3}$$

holds for the kinetic energy K of the hybrid system in its motion relative to the base.

In fact

$$K - \frac{M\dot{y}_c^2}{2} = \frac{\rho}{2}\left\{\int_0^l \dot{u}^2 dx - \frac{\rho}{M}\left(\int_0^l \dot{u}dx\right)^2\right\}$$

By the Cauchy–Bunyakovskii inequality, the expression in braces is non-negative and (2.3) follows from this.

We note that $E \geq F \geq M\dot{y}_c^2/2$. Hence

$$|\dot{y}_c| \leq \sqrt{2E/M} \tag{2.4}$$

According to (2.2), we obtain

$$dE/dt \leq |v(t)|\sqrt{2EM}$$

On integrating the last inequality, taking account of the null initial conditions and inequality (2.4), we have

$$|\dot{y}_c| \leq v_1(t), \quad v_1(t) = \int_0^t |v(t)|dt \tag{2.5}$$

In order to obtain a further preliminary estimate, we integrate equality (2.1) with respect to t. We obtain

$$M\int_0^t |\dot{y}_c||v(t)|dt \geq I(t), \quad I(t) = \int_0^t \dot{y}F(\dot{y})dt \tag{2.6}$$

We now estimate the integral on the left-hand side of (2.6), having made use of limit (2.5). On integrating the product $|v(t)|v_1(t)$ by parts, we shall have

$$\int_0^t |v(t)|v_1(t)dt = \frac{1}{2}\left(\int_0^t |v(t)|dt\right)^2$$

Taking account of (1.3), (2.5) and (2.6), we next obtain the required limit

$$I(t) \leq \frac{M}{2}J_0^2 \quad \forall v(\cdot)\in D \quad \forall t \geq 0 \tag{2.7}$$

3. We will now estimate the maximum displacement of the supporting body relative to the base. We first note that the equality

$$\int_0^\infty \dot{y}(t;v)F[\dot{y}(t;v)]dt = F_0\int_0^\infty |\dot{y}(t;v)|dt \tag{3.1}$$

holds according to the definition of dry friction.

At the same time

$$|y(t;v)| = \left|\int_0^t \dot{y}(t;v)dt\right| \leq \int_0^t |\dot{y}(t;v)|dt$$

and this means that

$$\sup_{t\in[0,\infty)} |y(t;v)| \leq \sup_{t\in[0,\infty)} \int_0^t |\dot{y}(t;v)|dt = \int_0^\infty |\dot{y}(t;v)|dt$$

Hence, by (2.7) and (3.1), we obtain the limit

$$\sup_{v(\cdot)\in D} S[v(\cdot)] \leq \frac{M}{2F_0}J_0^2 \tag{3.2}$$

This limit "works well" in two limiting cases: the mass of the elastic body is infinitesimal compared with the mass of the supporting body and the elastic body possesses a very high rigidity such that, in the limit, it may be treated as an absolutely rigid body. In both these cases, the hybrid system essentially degenerates into the single mass system considered previously in [2]. It was shown in [2] that the maximum possible displacement of a solid of mass M (here, either $M = m$ or $M = m + \rho l$) under the action of perturbations from class D is equal to $MJ_0^2/(2F_0)$ and is attained in the limit under the action of an instantaneous impact with the maximum permissible intensity J_0. We note that there is one further limiting case and this is, in fact, when the modulus of the elastic body is very small and limit (3.1) can turn out to be too high. In this case, the effect of the elastic body on the supporting solid will be small and we would therefore expect [2] inequality (3.2) to be replaced by the equality when $M = m$.

We will now attempt to estimate the functional $S[v(\cdot)]$, which should also take account of this limiting case. We transform the initial system of equations (1.1) and (1.2) and make the change of variables

$$\tau = t\nu,\ u = zL,\ y = \eta L,\ x = \theta l,\ \nu^2 = \sigma/(\rho l^2),\ L = F_0/(m\nu^2)$$

We obtain the system

$$\ddot{\eta} = -f(\dot{\eta}) + \varepsilon z'(0,\tau) + w(\tau),\ \ddot{z} - z'' = f(\dot{\eta}) - \varepsilon z'(0,\tau) \tag{3.3}$$

$$\eta(0) = \dot{\eta}(0) = z(\theta,0) = \dot{z}(\theta,0) = 0,\ z(0,\tau) = z'(1,\tau) = 0$$

$$(f = F/F_0,\ w = v/(L\nu^2) = mv/F_0,\ \varepsilon = \rho l/m)$$

Here, a dot denotes a derivative with respect to the dimensionless time τ while a prime denotes a derivative with respect to θ.

In this case, inequality (1.3) takes the form

$$\int_0^\infty |w(\tau)| d\tau \leqslant \beta_0,\ \beta_0 = \frac{\nu m J_0}{F_0} \tag{3.4}$$

We shall assign the piecewise-continuous functions $w(\tau)$, which satisfy inequality (3.4) and determine the perturbation in system (3.3), to the class D_*. We now plan the course of the subsequent arguments. We assume that it has been successfully shown that the inequality $|z'(0,\tau;w)| \leqslant a$, where a is a certain constant to be determined, is satisfied for any perturbations $w(\cdot) \in D_*$ and any $\tau \geqslant 0$. Then, if $1 - \varepsilon a > 0$, it is possible to obtain an estimate for $\sup_\tau |\eta(\tau;w)|$ (here, $\eta(\tau;w), z(0,\tau;w)$ is the solution of system (3.3) with the corresponding initial and boundary conditions). Actually, in this case, when account is taken of the first equation of (3.3) and the definition of the force due to dry friction, we have

$$\int_0^\tau |\dot{\eta}(\tau;w)||w(\tau)| d\tau \geqslant \int_0^\tau \dot{\eta}(\tau;w) p(\tau;w) d\tau \geqslant (1-\varepsilon a)\int_0^\tau |\dot{\eta}(\tau;w)| d\tau \tag{3.5}$$

$$(p(\tau;w) = f[\dot{\eta}(\tau;w)] - \varepsilon z'(0,\tau;w))$$

Using inequality (3.5) and then arguments analogous to those presented above, we obtain, after changing to the initial (dimensional) quantities

$$S[v(\cdot)] \leqslant \frac{mJ_0^2}{2F_0(1-\varepsilon a)} \tag{3.6}$$

It now remained to ascertain the value of the parameter a. As before, we shall assume that the inequalities

$$|z'(0,\tau;w)| \leqslant a\ \ \forall \tau \leqslant 0;\ 1 - \varepsilon a > 0 \tag{3.7}$$

hold.

It can be shown that the inequality

$$\int_0^\infty |p(\tau;w)| d\tau \leqslant \beta_0 \tag{3.8}$$

is satisfied, when account is taken of (3.7), for any perturbation $w(\cdot) \in D_*$.

In order to do this, we specify a certain arbitrary perturbation $w(\tau)$ from the class D_* and consider segments $[\tau_i, \tau_{i+1}]$ within which the velocity $\dot{\eta}(\tau;w) \neq 0$ and at the ends of which $\dot{\eta}(\tau_i;w) = \dot{\eta}(\tau_{i+1};w) = 0$. On integrating the first equation of (3.3) in these segments, allowing for assumption (3.7), we shall have

$$\int_{\tau_i}^{\tau_{i+1}} |p(\tau; w)| d\tau \leq \int_{\tau_i}^{\tau_{i+1}} |w(\tau)| d\tau \tag{3.9}$$

We now choose those intervals in which the velocity $\dot{\eta}(\tau_i; w) \equiv 0$ (these are possible segments of "prolonged arrest" of the supporting body on account of the existence of dry friction in the system). The equality $p(\tau; w) = w(\tau)$ holds in the case of such segments. Consequently, relation (3.9) holds with the equality sign. On successively summing inequality (3.9) and the above-mentioned equalities using (3.4), we finally obtain the required inequality (3.8). By virtue of the arbitrariness of the perturbation $w(\tau)$, (3.8) holds for any $w(\cdot) \in D$.

We now consider the second equation of (3.3) which we rewrite in the form

$$\ddot{z} - z'' = p(\tau; w), \quad z(\theta, 0) = \dot{z}(\theta, 0) = 0, \quad z(0, \tau) = z'(1, \tau) = 0$$

On representing the solution of this equation in the form of a series in the eigenfunctions of the corresponding homogeneous boundary-value problem, we transfer to the computational system of ordinary differential equations

$$\ddot{T}_n + \lambda_n^2 T_n = \sqrt{2} p(\tau; w) / \lambda_n, \quad T_n(0) = \dot{T}_n(0) = 0, \quad n \geq 1 \tag{3.10}$$

$$\lambda_n = \pi(2n - 1)/2$$

Hence

$$z(\theta, \tau; w) = \sum_{n=1}^{\infty} Q_n(\theta) T_n(\tau; w), \quad Q_n(\theta) = \sqrt{2} \sin(\lambda_n \theta)$$

The solution of system (3.1) can be represented in the following form

$$T_n(\tau; w) = \frac{\sqrt{2}}{\lambda_n} \int_0^t g_n(\tau - \chi) p(\chi; w) d\chi, \quad n \geq 1; \quad g_n(\tau) = \frac{\sin(\lambda_n \tau)}{\lambda_n}$$

When account is taken of the explicit expression for $z'(0, \tau; w)$ and, also, inequality (3.8), we have

$$|z'(0, \tau; w)| \leq 2\beta_0 \sup_{\tau \in [0, \infty)} r(\tau), \quad r(\tau) = \left| \sum_{n=1}^{\infty} g_n(\tau) \right|$$

This series converges by the Dirichlet criterion. We also note that the function $r(\tau)$ is periodic with period $\tau_0 = 2$ and we find numerically that $\max_\tau r(\tau) \approx 1/2$. So

$$a \approx \beta_0 = \frac{J_0 m}{F_0 l} \sqrt{\frac{\sigma}{\rho}} \tag{3.11}$$

We now jointly consider the limits (3.1) and (3.6) and elucidate under which conditions one of them is better than the other. By comparing $1+\varepsilon$ and $(1 - \varepsilon a)^{-1}$ with one another subject to the condition that $1 - \varepsilon a > 0$, we can write the single limit

$$S[v(\cdot)] \leq S^* \begin{cases} (1 - \varepsilon a)^{-1}, & a \leq (1+\varepsilon)^{-1} \\ 1 + \varepsilon, & a > (1+\varepsilon)^{-1}, \end{cases} \quad S^* = \frac{m J_0^2}{2 F_0}$$

where S^* plays the role of a certain "basic" quantity which corresponds to the maximum possible displacement of an isolated solid under the action of dry friction and a perturbation from class D.

In the easily checked limiting cases: $\varepsilon \to 0$ ($\rho l \ll m$, that is, in the case of an infinitesimal mass of the elastic body) and $a \to 0$ (in particular, according to (3.11), when $\sigma \to 0$, that is, in the case of an infinitesimal stiffness of the elastic body), the value of the limit tends to the base value S^* and is identical in the limit to the exact value of the maximum displacement S_0. There is one more limiting case when the elastic body possesses a very high stiffness: $a \to \infty$ ($\sigma \to \infty$). The hybrid system then degenerates into a rigid body of mass M and the limit of its displacement tends to the value $S^*(1 + \varepsilon)$.

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